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THE VELOCITY OF PROPAGATION OF A SIGNAL IN A FLUID WITH RELAXATION*

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Certain mathematical properties of a model of fluid hydrodynamics are investigated taking the effects of memory (lag) in transport phenomena into account. Relaxation kernels are introduced in the transport equation instead of the kinetic coefficients (viscosity and thermal conductivity) in such models. Rheological relationships algebraic in form are replaced by integral relations in time. The principal result is the derivation of constraints that are imposed on certain parameters of the integral kernels when necessary if the condition of finiteness of the propagation is taken in a small perturbation medium. A general scheme is constructed for estimating the relaxation kernels of a coupled system of hydrodynamics equations with a memory.

To ensure the finiteness of the velocity of propagation of small perturbations in a fluid the Navier-Stokes-Fourier material relationships must be modified. It was proposed in /1/ to eliminate the paradox of instantaneous signal propagation during heat conduction by replacing the parabolic equation for the temperature by a hyperbolic equation. It is shown /2/ that finiteness of the signal rate for transport processes is associated with the relaxation nature of the flux dependence on the gradient of the transferable quantity. The need to replace the kinetic coefficients by relaxation kernels is known from non-equilibrium thermodynamics /3, 4/ and experiment /5/.

1. We will study the motion of a fluid in an infinite space with respect to a system of Cartesian coordinates. The subscripts i, j, k, l correspond to the coordinates and run through the values 1, 2, 3. Summation is over the repeated subscripts.

The fluid state at any time t is given by the field of the density ρ, the velocity v_i, the temperature T. The symmetric stress tensor in a viscous fluid is given by the expression $p_{ij} = -p\delta_{ij} + \tau_{ij}$, where $p = p(\rho, T)$ is the thermodynamic pressure, and τ_{ij} is the viscous stress tensor /6/. Let q_i be the heat flux vector in the fluid. Entropy production in the fluid particle is determined by the expression

$$\sigma = T^{-1}\tau_{ij}e_{ij} + q_i (T^{-1})_{,i}, \quad e_{ij} = {}^1/_2 (v_{i,j} + v_{j,i}) \tag{1.1}$$

The material relationships

$$\tau_{ij} = \eta_V\lambda\delta_{ij} + 2\eta_S s_{ij}, \quad \lambda = e_{kk}, \quad s_{ij} = e_{ij} - {}^1/_3\delta_{ij}\lambda \tag{1.2}$$

$$q_i = -\varkappa T_{,i} \tag{1.3}$$

hold in the Navier-Stokes-Fourier model.

Here η_V is the volume viscosity, η_S is the shear velocity, and $\varkappa$ is the thermal conductivity, which depend on ρ, T.

We will investigate small perturbations of the density, velocity, and temperature in the background of a homogeneous fluid at rest. Consequently, we assume that $\rho = \rho_0 + r$, $T = T_0 + \theta$, where r, v_i, θ are small and ρ_0, T_0 are constants.

Let us introduce auxiliary notation. Let $g = g(t)$ be a certain real function. Let the symbol $g_F(\omega)$ denote its Fourier transform

$$g_F(\omega) = \int\limits_{-\infty}^{+\infty} e^{-i\omega t} g(t)\, dt$$

The equality

$$(g_F(\omega))^* = g_F(-\omega^*) \tag{1.4}$$

hold because of $g(t)$ is real.

Later the Paley-Wiener theorem will be utilized in the following form (see the proof in /7/).

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A. Let E be a space of rapidly decreasing functions on $\mathbb{R}$, and E' its conjugate space. If $f \in E'$ and the support f lies in the interval $[J, +\infty)$, then $F = f_F$ is a holomorphic function in the domain $\operatorname{Im}\omega < 0$ that satisfies the inequality

$$| F(\omega) | < C (1 + |\omega|)^{N_1} | \operatorname{Im}\omega |^{-N_2} \exp (J \operatorname{Im}\omega) \tag{1.5}$$

for certain positive constants C, N_a. If f is a smooth function here, then the function F is continuous up to the line $\operatorname{Im}\omega = 0$.

B. Let $F = F(\omega)$ be a holomorphic function in the domain $\operatorname{Im}\omega < 0$ that satisfies the inequality (1.5) for certain J, C, N_a, where $C, N_a > 0$. Then a function $f \in E'$ exists with support in the interval $[J, +\infty)$ such that $f_F = F$.

In order to take account of the finiteness of the signal propagation velocity in a fluid, it is natural to change from the model (1.2) and (1.3) to a circular model in which the viscosity and thermal conductivities are replaced by certain relaxation kernel $K_a = K_a(t)$ $(a = 1, 2, 3)$. In the linear approximation we obtain ($g_1 * g_2$ is the convolution of the functions g_1 and g_2)

$$\tau_{ij} = K_1 * \lambda\delta_{ij} + 2K_2 * s_{ij}, \quad q_i = -K_3 * \theta_{,i} \tag{1.6}$$

For slow processes the relationships (1.6) change to (1.2) and (1.3) where

$$\eta_V(\rho_0, T_0) = \int_{-\infty}^{+\infty} K_1(t)\,dt, \quad \eta_S(\rho_0, T_0) = \int_{-\infty}^{+\infty} K_2(t)\,dt, \quad \varkappa(\rho_0, T_0) = \int_{-\infty}^{+\infty} K_3(t)\,dt$$

We investigate the conditions that the kernels $K_a(t)$ satisfy. It follows from the causality principle that $K_a(t) = 0$ for $t < 0$. We will assume (and this is verified sufficiently well by statistical physics) that $K_a(t)$ are smooth positive monotonic rapidly decreasing functions for $t \geqslant 0$. According to Theorem A $K_{aF}(\omega)$ $(a = 1, 2, 3)$ are holomorphic functions in the domain $\operatorname{Im}\omega < 0$, and are continuous down to the line $\operatorname{Im}\omega = 0$.

It follows from the second law of thermodynamics that entropy production in a fluid particle is non-negative

$$W = \int_{-\infty}^{+\infty} \sigma\,dt \geqslant 0 \tag{1.7}$$

for a process for which r, v_i, θ tend to zero as $t \to \pm\infty$.

In the lowest order of smallness we obtain from (1.1)

$$W(x^l) = \frac{1}{T_0} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\tau \Big[K_1(t-\tau)\lambda(t, x^l)\lambda(\tau, x^l) + 2K_2(t-\tau) s_{ij}(t, x^l) s_{ij}(\tau, x^l) + \frac{1}{T_0} K_3(t-\tau)\theta_{,i}(t, x^l)\theta_{,i}(\tau, x^l) \Big]$$

Changing to Fourier transforms here, taking (1.4) into account, we have

$$W(x^l) = \frac{1}{\pi T_0} \int_0^{+\infty} d\omega \Big[\operatorname{Re} K_{1F}(\omega) |\lambda_F(\omega, x^l)|^2 + 2 \operatorname{Re} K_{2F}(\omega) |s_{ijF}(\omega, x^l)|^2 + \frac{1}{T_0} \operatorname{Re} K_{3F}(\omega) |\theta_{,iF}(\omega, x^l)|^2 \Big] \tag{1.8}$$

By virtue of the arbitrariness of $\lambda_F(\omega, x^l)$, $s_{ijF}(\omega, x^l)$ and $\theta_{,iF}(\omega, x^l)$ for $\omega \geqslant 0$ inequalities that are compatibility conditions for the model (1.6) with thermodynamics $\operatorname{Re} K_{aF}(\omega) \geqslant 0$ $(a = 1, 2, 3)$, $\omega \in \mathbb{R}$ follow from the relationships (1.7) and (1.8). We will use stronger inequalities that are apparently always satisfied in practice

$$\operatorname{Re} K_{aF}(\omega) > 0 \quad (a = 1, 2, 3) \tag{1.9}$$

where ω is an arbitrary real number. It follows from the assumptions made relative to the kernels and the general properties of holomorphic functions that the inequalities (1.9) are satisfied in the whole lower half-plane of the complex plane $\operatorname{Im}\omega \leqslant 0$.

The asymptotic form

$$K_{aF}(\omega) = -i\omega^{-1} K_a(0) + o(|\omega|^{-1}), \quad \operatorname{Im}\omega \leqslant 0 \tag{1.10}$$

results from the smoothness properties of the kernels.

2. Fluid motion is described by the dynamic continuity, momentum, and energy equations

$$\frac{d}{dt}\rho = -\rho v_{i,i}, \qquad \frac{d}{dt} v_i = \rho^{-1} p_{ij,j} + f_i \tag{2.1}$$
$$\frac{d}{dt}\left(\frac{1}{2} v_i v_i + U\right) = \rho^{-1}(p_{ij} v_j)_{,i} - \rho^{-1} q_{i,i} + \varepsilon + f_i v_i$$

Here $U = U(\rho, T)$ is the fluid internal energy per unit mass, f_i are the external mass forces, and ε is the heat source per unit mass. Let us introduce the entropy per unit mass $S = S(\rho, T)$. The entropy S is connected with the energy U by the differential relationship

$$dU = TdS - pd(1/\rho) \tag{2.2}$$

We will denote the thermodynamic quantity evaluated for $\rho = \rho_0$, $T = T_0$ by the symbol for this quantity with a zero subscript. Then in a linear approximation we obtain from (2.1) and (2.2)

$$\partial r/\partial t = -\rho_0 v_{i,i} \tag{2.3}$$
$$\partial v_i/\partial t = \rho_0^{-1}[-(p_{\rho 0} r + p_{T0}\theta)\delta_{ij} + \tau_{ij}]_{,j} + f_i$$
$$\partial\theta/\partial t = -p_{T0}\rho_0^{-1} S_{T0}^{-1} v_{i,i} - \rho_0^{-1} U_{T0}^{-1} q_{i,i} + U_{T0}^{-1}\varepsilon$$

Expressions (1.6) must be used in (2.3) for τ_{ij}, q_i.
We introduce the auxiliary notation

$$K_0(t) = K_1(t) + {}^4/_3 K_2(t), \tag{2.4}$$
$$M_a = M_a(\omega) = \rho_0^{-1} K_{aF}(\omega) \ (a = 0, 1, 2), \ M_3 = M_3(\omega) =$$
$$\rho_0^{-1} U_{T0}^{-1} K_{3F}(\omega), \ \rho_0^{-1} K_a(0) = m_a \ (a = 0, 1, 2),$$
$$\rho_0^{-1} U_{T0}^{-1} K_3(0) = m_3, \ \alpha = p_{\rho 0}, \ \beta = p_{T0}^2 \rho_0^{-2} S_{T0}^{-1}$$

We will assume that $\alpha \geqslant 0$, $\beta \geqslant 0$.
To investigate signal propagation in a fluid, the solution of (2.3) with a δ-like source must be considered

$$f_i = f_{i0}\delta(t)\,\delta(x^1)\,\delta(x^2)\,\delta(x^3), \ \varepsilon = \varepsilon_0\delta(t)\,\delta(x^1)\,\delta(x^2)\,\delta(x^3) \tag{2.5}$$

It is natural to solve system (2.3) with the source term (2.5) by the Fourier transform method in time and the space variable. We will denote the transposition operation of matrices by the symbol "+". We introduce the column matrix k (wave vector) and l by the relationships $k^+ = \| k_1 k_2 k_3 \|$, $l^+ = \| 0 f_{i0} U_{T0}^{-1}\varepsilon_0 \|$. Let $k^2 = k^+ k$ and let us select a point of space with the coordinates $x_0^i = L\delta_1^i$, $L > 0$. Then the Fourier transform of the density, velocity, and temperature fields in time at this point of space will be

$$\left\| \begin{matrix} r_F(\omega, x_0^j) \\ v_{iF}(\omega, x_0^j) \\ \theta_F(\omega, x_0^j) \end{matrix} \right\| = \int_{-\infty}^{+\infty} e^{-i\omega t} \left\| \begin{matrix} r(t, x_0^j) \\ v_i(t, x_0^j) \\ \theta(t, x_0^j) \end{matrix} \right\| dt = Yl \tag{2.6}$$

$$Y = \frac{1}{(2\pi)^3}\int e^{ik_1 L}(\Sigma(\omega, k_l))^{-1}\, dk_1\, dk_2\, dk_3 \tag{2.7}$$

$$\Sigma(\omega, k_l) = i\omega 1 + \left\| \begin{matrix} 0 & i\rho_0 k^+ & 0 \\ i\rho_0^{-1}\alpha k & (M_1 + {}^1/_3 M_2) kk^+ + M_2 k^2 & i\rho_0^{-1} p_{T0} k \\ 0 & i p_{T0}\rho_0^{-1} S_{T0}^{-1} k^+ & M_3 k^2 \end{matrix} \right\|$$

$$\det \Sigma(\omega, k_l) = (i\omega + k^2 M_2)^2 [k^4 M_3 (i\omega M_0 + \alpha) +$$
$$ik^2\omega(\alpha + \beta + i\omega(M_0 + M_3)) - i\omega^3] = P(\omega, k^2)$$

It is convenient to go over to integration in spherical coordinates R, ψ, φ, related to k_1, k_2, k_3 by the equalities

$$k_1 = R\sin\psi\cos\varphi, \ k_2 = R\sin\psi\sin\varphi, \ k_3 = R\cos\psi$$
$$R \geqslant 0, \ \psi \in [-\pi/2, \pi/2], \ \varphi \in [-\pi, \pi]$$

in expression (2.7).
After integration with respect to φ each matrix element is the sum of components of the form

$$y_\alpha = c_\alpha \int e^{iRL\sin\psi} R^{n+2} \sin^l\psi \cos^{m+1}\varphi\, (P(\omega, R^2))^{-1}\, dR\, d\psi \tag{2.8}$$

$$0 \leqslant R < +\infty, \ -\pi/2 \leqslant \psi \leqslant \pi/2$$

where n, l, m are certain natural numbers $c_\alpha = c_\alpha(\omega)$.

We note that the equation $P(\omega, R^2) = 0$ in R has six roots, where the set of roots is invariant relative to the inversion transformation $R \mapsto (-R)$. It is convenient to number these roots so that $\pm R_1(\omega)$ are roots of the equation

$$i\omega + R^2 M_2 = 0 \tag{2.9}$$

where $\operatorname{Im} R_1 \geqslant 0$, and $\pm R_2(\omega), \pm R_3(\omega)$ are roots of the equation

$$R^4 M_3 (i\omega M_0 + \alpha) + iR^2\omega(\alpha + \beta + i\omega(M_0 + M_3)) - i\omega^3 = 0 \tag{2.10}$$

where $\operatorname{Im} R_a(\omega) \geqslant 0 \ (a = 2, 3)$.

Lemma. The strict inequalities $\operatorname{Im} R_a(\omega) > 0 \ (a = 1, 2, 3)$ hold for all ω lying in the lower complex half-plane.

Proof. We assume that $\operatorname{Im} R_1(\omega) = 0$ for a certain ω. Then from (2.9) we obtain $\operatorname{Re} M_2 \leqslant 0$, which contradicts (1.9).

We now investigate (2.10). The quantity $z = -i\omega/R^2$ satisfies the equation resulting from (2.10)

$$z^2 - (\lambda_1 + \lambda_2 + \lambda_3) z + \lambda_1\lambda_2 = 0 \tag{2.11}$$
$$\lambda_1 = M_0 + \alpha/(i\omega), \ \lambda_2 = M_3, \ \lambda_3 = \beta/(i\omega)$$

Let $\alpha_j = \operatorname{Re}\lambda_j, \beta_j = \operatorname{Im}\lambda_j \ (j = 1, 2, 3)$. We note that according to the assumptions made earlier

$$\alpha_j > 0 \ (j = 1, 2); \ \alpha_3 \geqslant 0; \ \operatorname{sign}\beta_j = -\operatorname{sign}\operatorname{Re}\omega \tag{2.12}$$

We consider the parameter β as variable, changing from zero to $+\infty$. Eq.(2.11) has two roots which depend continuously on β: $z_a = z_a(\beta) \ (a = 1, 2)$, where for all β, α

$$\operatorname{Re} z_a(\beta) > 0 \tag{2.13}$$

Indeed, for $\beta = 0$ we have $z_a(0) = \lambda_a \ (a = 1, 2)$ and $\operatorname{Re} z_a(0) = \alpha_a > 0$ by virtue of (2.12). For a certain positive β and a certain a let $\operatorname{Re} z_a(\beta) = 0$. Substituting this root into the left-hand side of (2.11) and taking the real and imaginary parts, two real equations can be obtained for the unknown quantity $\operatorname{Im} z_a(\beta)$. It is possible to eliminate $\operatorname{Im} z(\beta)$ from these equations and then the equality

$$0 = -(\alpha_1\beta_2 + \alpha_2\beta_1)^2 + \sum_{j=1}^{3}\beta_j \sum_{i=1}^{3}\alpha_i(\alpha_1\beta_2 + \alpha_2\beta_1) + (\alpha_1\alpha_2 - \beta_1\beta_2)\left(\sum_{j=1}^{3}\alpha_j\right)^2$$

that is incompatible with (2.12) is obtained.

Now let $\operatorname{Im} R_a(\omega) = 0$ for certain ω where $a = 2$ or $a = 3$. Then

$$\operatorname{Re} z_{a-1} = \operatorname{Im}\omega/R_a^2 \leqslant 0$$

which contradicts (2.13). The lemma is proved.

We return to the investigation of (2.8). We note that since the elements of the matrices $P(\omega, k^2)(\Sigma(\omega, k_l))^{-1}$ depend as polynomials on k_l, then the equality $n = l + m$ holds. Furthermore, the matrix $\Sigma(\omega, k_l)$ remains invariant under the formal substitution $(R, \psi, \varphi) \mapsto (-R, -\psi, -\varphi)$, whereupon it follows that $m = 2q$, where q is a natural number.

We substitute $u = \sin\psi$ into (2.8) and integrate with respect to the variable u

$$y_\alpha = c_\alpha \sum_{d=0}^{q} C_q{}^d (-1)^d \int\limits_{0 \leqslant R < +\infty, \ -1 \leqslant u \leqslant 1} \exp(iRLu) R^{l+2q+2} u^{l+2d} (P(\omega, R^2))^{-1} dR\, du =$$
$$c_\alpha \sum_{d=0}^{q} a_d \int\limits_{-\infty}^{+\infty} R^{2(q-d)+1} \exp(iRL)(P(\omega, R^2))^{-1} dR$$

Here C_q^d are binomial coefficients, and a_d are certain constants. Integration with repect to R can be performed in the last integral by using the theory of residues and then the expression

$$y_\alpha = \sum_{a=1}^{3} s_{\alpha a}(\omega) \exp(iR_a(\omega) L) \tag{2.14}$$

is obtained where $s_{\alpha a}(\omega)$ are holomorphic functions in the half-plane $\operatorname{Im}\omega < 0$ that satisfy the inequality

$$| s_{\alpha a}(\omega) | \leqslant C \mid \operatorname{Im} \omega \mid^{-N_1} (1 + | \omega |)^{N_2} \tag{2.15}$$

for certain constants $C, N_a > 0$.

It follows from (2.14) and (2.6) that

$$\left\| \begin{matrix} r_F(\omega, x_0^j) \\ v_{iF}(\omega, x_0^j) \\ \theta_F(\omega, x_0^j) \end{matrix} \right\| = \sum_{a=1}^{3} l_a \exp(iLR_a(\omega))$$

where $l_a = l_a(\omega)$ are vectors whose components satisfy the inequality (2.15).

Let us introduce the functions

$$L_a = L_a(\omega) = \operatorname{Im} R_a(\omega)/(-\operatorname{Im} \omega), \ \operatorname{Im} \omega < 0 \ (a = 1, 2, 3)$$

into the considerations.

For the signal velocity in the fluid not to exceed a certain a priori given velocity c, according to Theorem A it is necessary and according to Theorem B it is sufficient that the inequalities

$$L_a(\omega) \geqslant c^{-1}, \ \operatorname{Im} \omega < 0 \ (a = 1, 2, 3)$$

be satisfied.

We set

$$V_a^{-1} = \inf_{\operatorname{Im} \omega < 0} L_a(\omega) \quad (a = 1, 2, 3)$$

We note that the functions $L_a = L_a(\omega)$ cannot reach the lower boundary at any point of the lower complex half-plane.

In fact, assume the opposite: let the point $\omega = \omega_0, \operatorname{Im} \omega_0 < 0$ be the point of the absolute minimum of the function $L_a(\omega)$ for certain a: $L_a(\omega_0) = V_a^{-1}$. Then the harmonic function of the two real variables ω_1 and ω_2

$$h_a = h_a(\omega_1, \omega_2) = \operatorname{Im} R_a(\omega_1 + \omega_2) + V_a^{-1} \omega_2$$

reaches the absolute minimum, equal to zero, at ω_0. Consequently, $h_a \equiv 0$, which however contradicts the continuity of the function $K_{aF}(\omega)$ on the real axis.

Thus the equality

$$V_a^{-1} = \lim_{\xi \to +\infty} \inf_{|\omega| \geqslant \xi} L_a(\omega)$$

holds.

Using (1.10) and (2.4), we now obtain

$$V_1 = m_2^{1/2}, \ V_{2,3} = (2m_3\alpha^\circ)^{1/2} [\alpha^\circ + \beta^\circ \pm ((\alpha^\circ + \beta^\circ) - 4m_3\alpha^\circ)^{1/2}]^{-1/2} \tag{2.16}$$
$$\alpha^\circ = \alpha + m_1, \ \beta^\circ = \beta + m_3$$

It is clear from the previous discussion that $V_a \ (a = 1, 2, 3)$ are the maximum propagation velocities of different modes in the fluid. The first equality in (2.16) corresponds to that obtained earlier /2/ since this mode describes vortex transport in a fluid.

The condition

$$V_a \leqslant c \ (a = 1, 2, 3) \tag{2.17}$$

must be taken into account in constructing fluid models with a finite signal propagation velocity. Together with (2.16) the inequalities (2.17) are constraints on the relaxation kernels.

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FIRST AND SECOND FUNDAMENTAL AXISYMMETRIC PROBLEMS OF ELASTICITY THEORY FOR DOUBLY-CONNECTED DOMAINS BOUNDED BY THE SURFACES OF A SPHERE AND A SPHEROID*

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The method of constructing solutions of the fundamental boundary-value problems for a homogeneous Lamé equation of multiconnected domains bounded by canonical surfaces of cylindrical and spheroidal coordinate systems described in /1/ is extended to domains with other geometry. The problems under consideration reduce to infinite systems of linear algebraic equations of the second kind with completely continuous opertors. A solution in the form of expansions in a small parameter for the problem of the hydrostatic pressure of a sphere with a centrally located spheroidal cavity is presented as an example.

1. We consider the first and second fundamental axisymmetric problems for a homogeneous Lamé equation

$$\nabla^2 \mathbf{u} + (1-2\nu)^{-1} \operatorname{grad} \operatorname{div} \mathbf{u} = 0 \tag{1.1}$$

(ν is Poisson's ratio) for a sphere with a spheroidal cavity whose axis passes through the centre of the sphere. Introducing identically directed systems of spherical coordinates (r, θ, φ) and prolate spheroidal coordinates (ξ_1, η_1, φ) superposed on the centres of the boundary surfaces, we obtain the following relation between the coordinates

$$r\cos\theta = c \operatorname{ch}\xi_1 \cos\eta_1 + a, \quad r\sin\theta = c \operatorname{sh}\xi_1 \sin\eta_1 \tag{1.2}$$

($2c$ is the focal length of the spheroidal system of coordinates, and a is the spacing between the centres of the boundary surfaces).

Let displacement vectors be given on the boundary

$$\mathbf{U}_{|r=R} = \sum_{k=0}^{\infty} [A_{k,1}^{(1)} P_k^{(1)}(\cos\theta)\,\mathbf{e}_\rho + A_{k,1}^{(2)} P_k(\cos\theta)\,\mathbf{e}_z] \tag{1.3}$$

$$\mathbf{U}_{|\xi_1=\xi_0} = \sum_{k=0}^{\infty} [A_{k,2}^{(1)} P_k^{(1)}(\cos\eta_1)\,\mathbf{e}_\rho + A_{k,2}^{(2)} P_k(\cos\eta_1)\,\mathbf{e}_z]$$

(e_ρ, e_z are unit vectors of the cylindrical system of coordinates). We later assume that

$$\sum_{k=0}^{\infty} k(k|A_{k,i}^{(1)}| + |A_{k,i}^{(2)}|) < \infty \quad (i=1,2) \tag{1.4}$$

We will seek the solution of problem (1.1) and (1.3) in the form

$$\mathbf{U} = \sum_{k=1}^{2}\sum_{n=0}^{\infty}\left[a_n^{(k)} \frac{n!}{R^n} \mathbf{W}_{k,n}^-(r,\theta) + \frac{a_n^{(k+2)}}{Q_n(\operatorname{ch}\xi_0)} \mathbf{U}_{k,n}^+(\xi_1,\eta_1)\right] \tag{1.5}$$

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